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CONTROL OF A PREDATOR–PREY SYSTEM WITH INTRASPECIES COMPETITION†

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The time-optimal control problem for a predator–prey system with intraspecies competition among the preys is considered. The controls are pesticides or insecticides that act on the preys or the predators. The optimal control is synthesized and the dependence of the response time on the parameters of the problem is analysed. The time-optimal control problem for a predator–prey system without intraspecies competition has been previously studied for the Lotka–Volterra model [1] and for the Mono model [2].

1. STATEMENT OF THE PROBLEM

CONSIDER a controlled system modelling the interaction of two populations with intraspecies competition [3]

$$X_1^{\cdot}(\tau) = (a_1 - a_2Y_1(\tau) - a_5X_1(\tau) - a_6u_1(\tau))\, X_1(\tau)$$
$$Y_1^{\cdot}(\tau) = (a_3X_1(\tau) - a_4 - a_7Y_1(\tau) - a_8u_1(\tau))\, Y_1(\tau)$$

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Here $X_1(\tau)$ and $Y_1(\tau)$ are the size of the prey population and the size of the predator population at a time τ, $u_1(\tau)$ is the control and $a_i>0$ are some constants characterizing the variation of the population size where a_2 and a_3 characterize the interspecies interaction, a_5 and a_7 characterize the intraspecies competition and a_6 and a_8 allow for the effect of the control on population size.

We introduce dimensionless variables $X(t)$, $Y(t)$, $u(t)$ and t defined by the formulas

$$X_1(\tau) = a_4 a_3^{-1} X(t),\quad Y_1(\tau) = a_1 a_2^{-1} Y(t),\quad u_1(\tau) = a_1 a_6^{-1} u(t),\quad \tau = a_1^{-1} t$$
$$b = a_4 a_1^{-1},\quad \alpha = (a_1 a_3)^{-1} a_4 a_5,\quad c = (a_2 a_4)^{-1} a_7 a_1,\quad d = (a_4 a_6)^{-1} a_8 a_1$$

In dimensionless variables, assuming that the predator population is without intraspecies competition ($c = 0$), the equations of the controlled system take the form

$$\begin{aligned} X^{\cdot}(t) &= (1 - Y(t) - \alpha X(t) - u(t))\, X(t) \\ Y^{\cdot}(t) &= b\,(X(t) - 1 - du(t))\, Y(t) \end{aligned} \tag{1.1}$$

The initial conditions at time $t_0 = 0$ are given by

$$X(0) = X_0,\ Y(0) = Y_0,\ X_0 > 0,\ Y_0 > 0 \tag{1.2}$$

The controls $u(t)$ are constrained by

$$0 \leqslant u(t) \leqslant \gamma \tag{1.3}$$

where γ is a given constant.

The control problem is to take system (1.1) from state (X_0, Y_0) to a nonzero equilibrium with $u = 0$ in a minimum time $T(X_0, Y_0, u_0)$, where u_0 is the optimal control.

2. TIME-OPTIMAL CONTROL WITH CONTROL AFFECTING THE PREYS

Consider system (1.1) assuming that the control acts only on the preys, i.e. $d = 0$. Then (1.1) takes the form

$$X^{\cdot} = (1 - Y - \alpha X - u)\, X,\ Y^{\cdot} = b\,(X - 1)\, Y \tag{2.1}$$

with initial conditions (1.2) and control constraints (1.3).

If we make the natural assumption that the prey population is greater in the absence of predators than in their presence, we obtain $0 \leqslant \alpha \leqslant 1$.

Note that for any measurable control $u(t)$ satisfying (1.3), the solution of problem (2.1), (1.2) is positive for any finite t. For $u = 0$, system (2.1) has at least one equilibrium inside the first quadrant—the point $R = (1, 1 - \alpha)$.

The time-optimal control problem involves taking the system (2.1) from state (X_0, Y_0) to state R in the shortest possible time. We denote by $T(X_0, Y_0, u)$ the first arrival of system (2.1), (1.2) and (1.3) at the point R under the control u. We thus conclude that the optimal control u_0 is determined by the relationship

$$\inf_u T(X_0, Y_0, u) = T(X_0, Y_0, u_0) \tag{2.2}$$

The rest of the analysis depends on the parameters of the problem.

The equilibrium is a stable focus. The existence of a stable focus corresponds to parameters satisfying the inequality

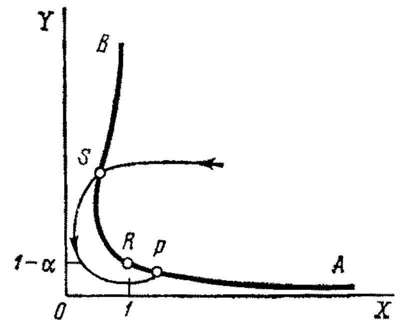

FIG. 1.

$$\alpha^2 < 4b(1 - \alpha) \tag{2.3}$$

Note that for $u = 0$ the stable focus R is unreachable in any finite time.

We will show that when inequality (2.3) is satisfied, an optimal control exists. Change to reverse time $t \to -t$. In reverse time, Eqs (2.1) take the form

$$X^{\cdot} = (Y + \alpha X + u - 1)X, \quad Y^{\cdot} = b(1 - X)Y \tag{2.4}$$

Denote by APR the part of the trajectory $X(t)$, $Y(t)$, $t \geqslant 0$ of system (2.4) with control $u = \gamma$ and initial conditions $X(0) = 1$, $Y(0) = 1 - \alpha$ such that $X(t) \geqslant 1$, $t \geqslant 0$ (Fig. 1).

Take the control $u = \gamma$ on APR and $u = 0$ everywhere outside APR. Since R is a stable focus, then with this control u system (2.1) will reach the point R in a finite time for any initial condition (1.2). This and [4] prove the existence of the optimal control u_0. Thus, by the maximum principle, a nonzero solution ψ_1, ψ_2 of the control system

$$\psi_1^{\cdot} = \psi_1(Y - 1 + u + 2\alpha X) - b\psi_2 Y, \quad \psi_2^{\cdot} = \psi_1 X + b\psi_2(1 - X) \tag{2.5}$$

exists such that [5]

$$\max_u H(X, Y, \psi_1, \psi_2, u) = H(X, Y, \psi_1, \psi_2, u_0) \equiv 0 \tag{2.6}$$

Here

$$H = \psi_1 ([1 - Y - u - \alpha X) X + \psi_2 b (X - 1) Y + \psi_0, \quad \psi_0 = \text{const} \leqslant 0$$

From Eqs (2.5) it follows that the functions $\psi_i(t)$ do not vanish in entire intervals. Indeed, if for instance $\psi_1(t) \equiv 0$ for $t \in [t_1, t_2]$, $t_1 < t_2$, then $\psi_1^{\cdot}(t) \equiv 0$ for $t \in [t_1, t_2]$. Therefore, by the first equation in (2.5) $\psi_2(t) \equiv 0$, $t_1 \leqslant t \leqslant t_2$, i.e. both conjugate variables ψ_1, ψ_2 vanish simultaneously, which is impossible. A similar argument shows that the functions ψ_1 and ψ_2 may only have simple zeros. Therefore, by (2.6), the optimal control u_0 is piecewise-constant, being either zero or γ.

For further analysis of the optimal control, it is helpful to change from the variables ψ_i to the variables φ_i by the formulas

$$\varphi_1(t) = X(t)\psi_1(t), \quad \varphi_2(t) = Y(t)\psi_2(t)$$

By the positivity of $X(t)$ and $Y(t)$, the signs of the functions φ_1 and φ_2 are equal to the signs of φ_1 and φ_2, respectively. From (2.1) and (2.5) we obtain the following equations for $\varphi_i^{\bullet}$:

$$\varphi_1^{\bullet}(t) = (-b\varphi_2 + \alpha\varphi_1)X, \quad \varphi_2^{\bullet}(t) = \varphi_1 Y \tag{2.7}$$

Partition the optimal trajectory into sections contained entirely either in the region $X>1$ or in the region $X<1$. Any section of the optimal trajectory contained entirely in the region $X>1$ may have at most one switching point, with the optimal control switching from $u=0$ to $u=\gamma$.

Indeed, let $X(t)>1$ for $t_1 \leqslant t \leqslant t_2$ and the control changes at the instant $\tau \in [t_1, t_2]$. Since τ is a switching point of the optimal control, then by (2.6) $\varphi_1(\tau)=0$ at this point. Now from (2.6) and (2.1) we obtain

$$\varphi_1 X^{\cdot}/X + \varphi_2 Y^{\cdot}/Y + \psi_0 = 0$$

Seeing that by (2.1) $Y^{\bullet}(t)>0$ for $t \in [t_1, t_2]$, we obtain the relationship

$$\varphi_2(t) = Y[-\psi_0 - \varphi_1 X^{\cdot}/X]/Y^{\cdot}, \quad t_1 \leqslant t \leqslant t_2$$

Now, by (2.7), we have the following equations for $t_1 \leqslant t \leqslant t_2$:

$$\varphi_1^{\cdot}(t) = \varphi_1(\alpha X + bYX^{\cdot}/Y^{\cdot}) + b\psi_0 XY/Y^{\cdot}, \quad \varphi_1(\tau) = 0 \tag{2.8}$$

If $\varphi_0 = 0$, then by (2.8) $\varphi_1(t) \equiv 0$, $t_1 \leqslant t \leqslant t_2$, which is impossible. But $\psi_0 \leqslant 0$. Therefore, the constant $\psi_0 < 0$, and so $\varphi_1(t)<0$ for $\tau < t \leqslant t_2$ and $\varphi_1(t)>0$ for $t_1 \leqslant t < \tau$. Therefore, applying the maximum principle, we conclude that as long as the optimal trajectory remains inside the region $X>1$, the optimal control is $u_0=\gamma$ for $t>\tau$ and $u_0=0$ for $t<\tau$. We similarly show that the section of the optimal trajectory entirely contained in the region $X<1$ may have at most one switching point, with the optimal control changing from $u=\gamma$ to $u=0$.

The qualitative behaviour of the switching curve depends on the ratio of the numbers α, b, γ.

First let

$$\gamma \geqslant 1 - \alpha - \alpha^2/(4b) \tag{2.9}$$

If inequalities (2.3) and (2.9) are satisfied, Eqs (2.1) for $u=\gamma$ either have no points of rest in the regions $X>0$, $Y>0$ or have one point of rest with the coordinates $(1, 1-\alpha-\gamma)$, which is a stable node. In either case, the section APR of the switching curve is a trajectory of system (2.1) for $u=\gamma$ (Fig. 1).

To obtain the curve APR, we can solve Eq. (2.4) for $u=\gamma$, $t>0$ with the initial condition $X(0)=1$, $Y(0)=1-\alpha$. The section RSB of the switching curve can be obtained numerically.

The numerical algorithm can be described as follows. Rewrite Eqs (2.7) in reverse time $t \to -t$

$$\varphi_1^{\cdot}(t) = (b\varphi_2 - \alpha\varphi_1)X, \quad \varphi_2^{\cdot}(t) = -\varphi_1 Y, \quad t>0 \tag{2.10}$$

Consider system (2.4) and (2.10) for $u=\gamma$ with the initial conditions

$$X(0) = 1, \quad Y(0) = 1 - a, \quad \varphi_1(0) < 0, \quad \varphi_2(0) \tag{2.11}$$

where $\varphi_1(0)$, $\varphi_2(0)$ are some given quantities. Solve problem (2.4), (2.10) and (2.11) for $u=\gamma$ in the interval $[0, \tau_1]$, where τ_1 is the first zero of the function $\varphi_1(t)$. At time τ_1, the control switches to zero. Then the solution of problem (2.4) and (2.10) is sought for $t \in [\tau_1, \tau_2]$, $u=0$, using the initial conditions at the point τ_1 as determined in the preceding step of the algorithm. Here τ_2 is the next zero of the function $\varphi_1(t)$ after τ_1. The point with coordinates $X(\tau_2)$ and $Y(\tau_2)$ lies on the switching curve RSB. To determine other points of the curve RSB, we need to run these procedures for other initial values of $\varphi_1(0)$ and $\varphi_2(0)$.

Figure 1 shows the switching curve $APRSB$ for parameter values $\gamma=2$, $\alpha=0.5$ and $b=1/2$, and Fig. 2 shows the Bellman function.

Now consider the case

$$0 < \gamma < 1 - \alpha - \alpha^2/(4b) \tag{2.12}$$

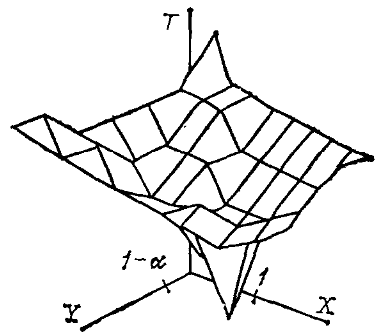

FIG. 2.

When inequalities (2.3) and (2.12) are satisfied, system (1.1) with $u = \gamma$ has two points of rest $R_1 = (1, 1-\alpha-\gamma)$ and $R_2 = (\alpha^{-1}(1-\gamma), 0)$, where R_1 is a stable focus and R_2 is a saddle point. Consider the separatrix in the system (2.1) with $u = \gamma$ which originates from the point R_2 and enters the point R_1. This separatrix leaves the saddle point R_2 with a slope coefficient $\alpha^{-1}(\gamma-1)(b+\alpha)<0$. Denote by Y_1 the maximum ordinate of the separatrix. By (2.1), the point of the separatrix where Y_1 is reached lies on the straight line $X = 1$. Given conditions (2.3) and (2.12), the optimal control switching curve has the following form depending on the value of Y_1. If $Y_1 < 1-\alpha$, then the qualitative behaviour of the switching curve is the same as in case (2.3) and (2.9) (Fig. 1). If $Y_1 = 1-\alpha$, the switching curve is as shown in Fig. 3 (for parameter values $\alpha = 0.03$, $\gamma = 0.4884$, $b = 1/16$). Here $R_2A\mathbf{P}R$ is the part of the separatrix of system (2.1) with $u = \gamma$ which leaves the point R_2 and enters the point R_1. Finally, for $Y_1 > 1-\alpha$, the switching curve is shown in Fig. 4 (for parameter values $\alpha = 0.03$, $\gamma = 0.02$ and $b = 1/16$), where $\mathbf{P}R$ is the trajectory of system (2.4) for $u = \gamma$ and initial conditions $X(0) = 1$, $Y(0) = 1-\alpha$. The section RSB in all cases, and in the last case the section $A\mathbf{P}$ also, were generated numerically (using the numerical algorithm described above). However, unlike the situations described in Figs 1–3, where the number of switching points is at most two, the number of switching points in Fig. 4 depends on the initial position of system (2.1) and may be greater than two.

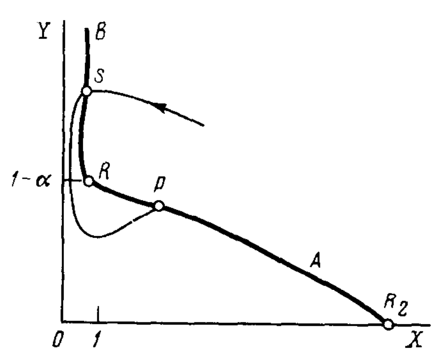

FIG. 3.

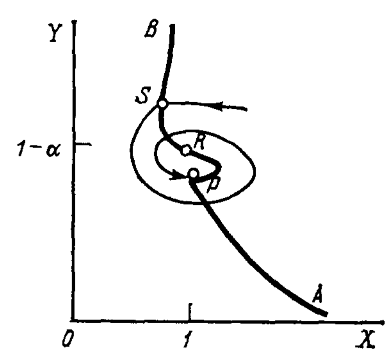

FIG. 4.

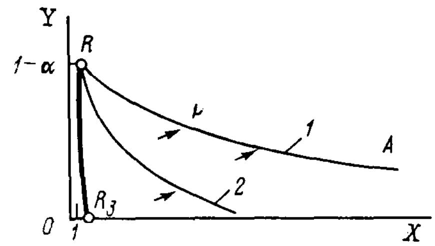

FIG. 5.

The equilibrium is a stable node. Here we consider parameters corresponding to the inequality

$$\alpha^2 \geqslant 4b\,(1-\alpha) \tag{2.13}$$

for which R is a stable node.

Note that for $u=0$ the stable node $R=(1, 1-\alpha)$ is unreachable in any finite time. This point has two entry directions for $\alpha^2>4b(1-\alpha)$ and one exist direction for $\alpha^2=4b(1-\alpha)$ (we do not distinguish between opposite entry directions having the same slope coefficient).

Denote by $R_3=(\alpha^{-1}, 0)$ the point of rest of system (2.1) for $u=0$ and by R_3R the separatrix that leaves the saddle point R_3 with the slope coefficient $b(1-\alpha^{-1})-1<0$ and enters the node R.

Now let APR be the trajectory of system (2.4) for $u=\gamma$, $t>0$ with the initial conditions $X(0)=1$, $Y(0)=1-\alpha$.

Note that system (2.1) for $u=\gamma$ does not have singular points inside the first quadrant if $\gamma\geqslant 1-\alpha$ and has one singular point R_1 (a stable node) if $\gamma<1-\alpha$. Analysis of the phase portrait of system (2.1) for $u=0$ and $u=\gamma$ shows that the controllability region D of this system lies between the curves R_3R and APR (Fig. 5) obtained for the following conditions: (1) $\alpha=0.5$, $b=1/4$, $\gamma=0.02$ for curve 1; (2) $\alpha=0.5$, $b=1/4$, $\gamma=2$ for curve 2. Here R_3R is not contained in D, and $APR\in D$. The optimal control is zero everywhere in D, except the curve APR, where it equals γ. For the points (X_0, Y_0) outside D, R is unreachable with any admissible control.

3. TIME-OPTIMAL CONTROL WITH CONTROL AFFECTING THE PREDATORS

Let us consider the time-optimal control problem assuming that the control acts directly only on the predators.

The equations of the system in dimensionless variables take the form

$$\begin{gathered} X^{\cdot} = (1-Y-\alpha X)\,X \\ Y^{\cdot} = b\,(X-1)\,Y - uY, \quad 0 \leqslant u \leqslant \gamma, \quad 0<\alpha<1, \quad \gamma>0 \end{gathered} \tag{3.1}$$

It is required to take the system from position (1.2) to the equilibrium $R=(1, 1-\alpha)$ in the least possible time. As before, the synthesis of the optimal control depends on the paramters of the problem. Assume that inequality (2.3) holds. Then, as in Sec. 2, we establish the existence of an optimal control and therefore prove relationships (2.6), where

$$\begin{gathered} H = \psi_1\,[(1-Y-\alpha X)\,X + \psi_2\,(b\,(X-1)-U)\,Y + \psi_0, \\ \psi_0 = \text{const} \leqslant 0 \end{gathered} \tag{3.2}$$

Here ψ_1, ψ_2 is some nonzero solution of the system of equations

$$\psi_1^{\cdot} = \psi_1 (Y - 1 + 2\alpha X) - b\psi_2 Y$$
$$\psi_2^{\cdot} = -\psi_2 (b (X - 1) - U) + \psi_1 X \tag{3.3}$$

From these relationships and the maximum principle it follows that the functions ψ_1 and ψ_2 do not have multiple roots. Moreover, the variables $\varphi_1 = X\psi_1$ and $\varphi_2 = Y\psi_2$ satisfy Eqs (2.7) as before.

We find that any section of the optimal trajectory entirely contained in the region $1 - Y - \alpha X < 0$ may have at most one switching point, with the optimal control switching from $U = 0$ to $U = \gamma$.

Indeed, assume that for $t_1 \leqslant t \leqslant t_2$ the optimal trajectory of system (3.1) is in the region $1 - Y - \alpha X < 0$, where $X^{\cdot}(t) < 0$ and there is a switching point $\tau \in [t_1, t_2]$. By (2.6) and (3.2), we should have $\alpha_2(\tau) = 0$ at the instant τ, and moreover

$$\varphi_1^{\cdot}(t) = X(t)(-\psi_0 - \varphi_2(t) Y^{\cdot}(t)/Y(t)) / X^{\cdot}(t), \quad t_1 \leqslant t \leqslant t_2$$

This and (3.3) give

$$\varphi_2^{\cdot}(t) = [-\alpha_2(t) X(t) Y^{\cdot}(t) - \psi_0 Y(t) X(t)]/X^{\cdot}(t), \quad \varphi_2(\tau) = 0, \quad t_1 \leqslant t \leqslant t_2$$

The last equations show that $\psi_0 < 0$, while $\varphi_2(t) < 0$ for $\tau < t \leqslant t_2$ and $\psi_2(t) > 0$ for $t_1 \leqslant t < \tau$.

We similarly prove that any section of the optimal trajectory entirely contained in the region $1 - Y - \alpha X > 0$ may have at most one switching point, with the optimal control switching only from $U = \gamma$ to $U = 0$.

Let us now investigate the trajectory of system (3.1) with $U = \gamma$ that passes through the point R. In reverse time $(t \to -t)$ this trajectory is described by the relationships

$$X^{\cdot}(t) = (\alpha X + Y - 1) X, \quad Y^{\cdot}(t) = (\gamma + b - bX) Y \tag{3.4}$$
$$X(0) = 1, \quad Y(0) = 1 - \alpha, \quad t \geqslant 0$$

First let $\gamma \geqslant b(\alpha^{-1} - 1)$.

The solution $X(t)$ and $Y(t)$ of problem (3.4) is monotone increasing in the interval $[0, \tau]$, where τ is the first instant when $Y^{\cdot}(\tau) = 0$. By (3.4), we have $X(\tau) = \gamma b^{-1} + 1$. This and (3.5) give

$$X^{\cdot}(\tau) > \alpha\gamma b^{-1} + \alpha - 1 \geqslant 0 \tag{3.5}$$

Therefore $X^{\cdot}(t) > 0$, $t \leqslant 0$ and $Y^{\cdot}(s) < 0$, $s > \tau$. Thus, using (3.4)

$$X(t) \to \infty, \quad Y(t) \to \infty, \quad t \to \infty$$

When inequalities (2.3) and (3.5) are satisfied, the optimal control switching curve in problems (3.1) and (3.2) is shown in Fig. 6, where *APR* is the trajectory of system (3.4). The switching curve *RSB* has been obtained numerically using the algorithm of Sec. 2.

Let us establish the form of the optimal control switching curve, assuming as before that inequality (2.3) holds and taking

$$\gamma < b(\alpha^{-1} - 1) = \gamma_1 \tag{3.6}$$

When conditions (2.3) and (3.6) are satisfied, system (3.1) with $U = \gamma$ has a point of rest $R_4 = (1 + \gamma b^{-1}, 1 - \alpha - \alpha\gamma b^{-1})$ inside the first quadrant. Investigating in the usual way Eqs (3.1) linearized in the neighbourhood of R_4 for $U = \gamma$, we conclude that the point R_4 is either a stable node for $\gamma_0 \leqslant \gamma < \gamma_1$ or a stable focus for

$$\gamma < \gamma_0 = -b + b^2 (\alpha^2/4 + b\alpha)^{-1} < \gamma_1 \tag{3.7}$$

If $\gamma_0 \geqslant \gamma < \gamma_1$ and inequality (2.3) is satisfied, the qualitative behaviour of the switching curve is the same as in Fig. 6 with the parameters $\alpha = 0.5$, $\gamma = 0.7$, and $b = 1/2$.

Finally, consider the case (2.3) and (3.7). Let $R_5 = (\alpha^{-1}, 0)$ be a saddle point of system (3.1) for $U = \gamma$, and R_5R_4 the separatrix leaving the saddle point R_5 with the slope coefficient $(-1 + \gamma - \gamma_1)$

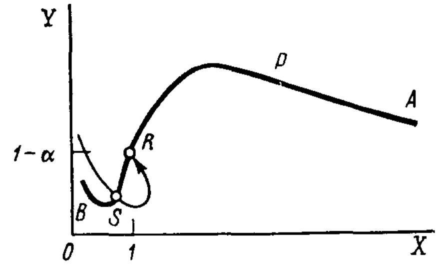

FIG. 6.

$(1+\gamma b^{-1})^{-1}<0$ and entering R_4. Denote by X_1 the minimum abscissa of this separatrix. The point of the separatrix where X_1 is achieved lies on the line $Y+\alpha X-1=0$.

When conditions (2.3) and (3.6) are satisfied, the optimal control switching curve can be described as follows:

1. If $X_1<1$, then the qualitative behaviour of the switching curve is as shown in Fig. 6 for the parameters $\alpha=0.5$, $\gamma=0.7$, $b=1/2$.
2. If $X_1=1$, then the switching curve is R_5**A**$PRSB$ (Fig. 7, obtained for $\alpha=0.03$, $\gamma=0.2619$ and $b=1/16$), where R_5**A**PR is the separatrix of system (3.1) with $U=\gamma$ that leaves the point R_5 and enters the point R_4; the section RSB is obtained numerically by the algorithm of Sec. 2.
3. If $X_1>1$, then the switching curve **A**$PRSB$ is shown in Fig. 8 for $\alpha=0.03$, $\gamma=0.02$ and $b=1/16$, where **P**R and RSB are generated numerically.

As in Sec. 2, consider the case when the parameters of system (3.1) satisfy condition (2.13).

System (3.1) for $U=\gamma$ does not have singular points inside the first quadrants if inequality (3.5) is satisfied; it has a stable node if (3.6) holds. From the phase portrait of system (3.1) for $U=0$ and $U=\gamma$ we see (Fig. 3) that the region of controllability D lies between the separatrix $R_3R\notin D$ of system (3.1) for $U=0$ and the trajectory $APR\in D$ of system (3.4), where $R_3=(\alpha^{-1},0)$. The optimal control is zero inside D and γ on APR.

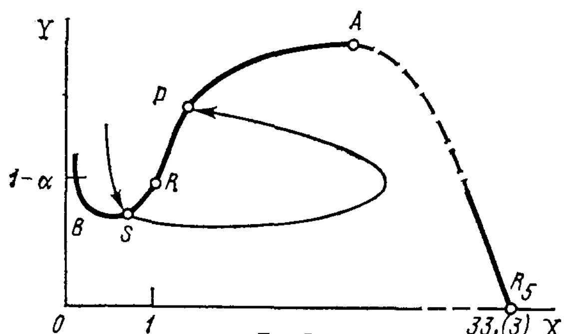

FIG. 7.

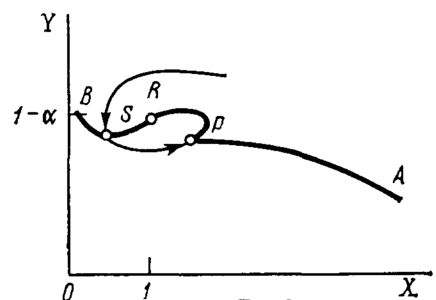

FIG. 8.

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